# On measures of nonnormality of matrices

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Abstract

This paper discusses measures $\mu_i$, $i = 1,\dots,9$, of nonnormality of matrices and their interrelations. Some of the measures are new. Each nonnegative measure $\mu_i(A)$ equals zero iff A is normal. We compare these measures with several new inequalities. Some of these comparisons, e.g., (C12), (C15) and (C16), manifest wellknown phenomena of ill conditioned eigenproblems.

## 1. Introduction

The class of nonnormal matrices has received some attention of numerical analysts. In particular in connection with certain eigenvalue algorithms normal and nonnormal matrices show quite different behaviour. Related to this fact is the difference in the sensitivity of the eigenvalues and eigenvectors under perturbations of the entries of the matrix.
For analyzing these difficulties several measures of nonnormality have appeared in the literature. We give here an overview of the measures used, introduce some new ones and give in particular comparisons between them.

Throughout this paper $n > 1$ is a fixed integer. Let $C^{n,n}$ denote the set of all complex $n \times n$ matrices and $A \in C^{n,n}$. We associate with A the following magnitudes.

- its eigenvalues $\lambda_i$ and its singular values $\sigma_i$, ordered such that

$$|\lambda_1| \geq |\lambda_2| \geq \dots |\lambda_n| \ , \quad \sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n;$$

- its polar factors $H_1, H_2$, i.e., the uniquely determined positive semi-definite square roots of $AA^*$ and $A^*A$;
- $\alpha_i$, $i = 1,\dots,n$, with $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n$, the eigenvalues of the hermitean part $(A+A^*)$ of $A$;
- $\beta_i$, $i = 1,\dots,n$, with $\beta_1 \geq \beta_2 \geq \dots \geq \beta_n$, the eigenvalues of the skew-hermitean part $(A-A^*)/(2i)$ of $A$.

$A$ is normal, if $A^*A - AA^* = 0$ holds. Let $N$ denote the set of all normal matrices in $C^{n,n}$, $U$ the set of all unitary matrices in $C^{n,n}$, and $D$ the set of diagonal matrices in $C^{n,n}$. We denote by $\|.\|_2$ the spectral norm and by $\|.\|_F$ the Frobenius norm of a matrix. For $X$ nonsingular $\kappa_i(X) := := \|X\|_i \ \|X^{-1}\|_i$, $i = 2,F$ is the condition number of $X$.

## 2. Characterization of normal matrices

There are quite a few charactizations for $A$ being normal.

Theorem 1. For $A \in C^{n,n}$ with eigenvalues $\{\lambda_i\}$, singular values $\{\sigma_i\}$, polar factors $H_i$ $(i = 1,2)$ and eigenvalues $\{\alpha_i\}$, $\{\beta_i\}$ of $(A+A^*)/2$ and $(A-A^*)/(2i)$ resp. as above the following are equivalent.

(i) $A$ is normal, i.e., $A^*A - AA^* = 0$;

(ii) There exists a $V \in U$ such that $V^*AV = \mathrm{diag}(\lambda_i)$;

(iii) $\|A\|_F^2 - \sum_{i=1}^{n} |\lambda_i|^2 = 0$;

(iv) $A = X\,\mathrm{diag}(\lambda_i)X^{-1}$ for some $X \in C^{n,n}$ nonsingular and $\kappa_2(X) = 1$;

(v) $|\lambda_i| - \sigma_i = 0, \quad i = 1,\dots,n$;

(vi) $H_1 = H_2$;

(vii) There exists a $V \in U$ such that both $V^* \frac{A+A^*}{2} V$ and $V^* \frac{A-A^*}{2} V$ are diagonal;

(viii) There exist permutations $p$ and $q$ of $\{1,\dots,n\}$ such that

$$\lambda_j = \alpha_{p(j)} + i\ \beta_{q(j)} \ , \quad j = 1,\dots,n. \ \square$$

We refrain from giving a proof, as the results are either trivial or easy consequences of the quantitative statements in theorem 2.

We make one exception. (viii) implies (iii) as can be seen from the simple equalities

$$\sum_{j=1}^{n} (\alpha_{p(j)}^2 + \beta_{q(j)}^2) = \left\|\frac{A+A^*}{2}\right\|_F^2 + \left\|\frac{A-A^*}{2i}\right\|_F^2 = \|A\|_F^2 .$$

## 3. Measures of Nonnormality of Matrices

Theorem 1 motivates the introduction of several measures of nonnormality.

The most natural measure of nonnormality seems to be

$$\mu_1(A) := \min\{\|A-N\|_F \mid N \in \mathcal{N}\},$$

and

$$\tilde{\mu}_1(A) := \min\{\|A-N\|_2 \mid N \in \mathcal{N}\}.$$

Besides this we consider

$$\mu_2(A) := \|A^*A - AA^*\|_F, \quad \tilde{\mu}_2(A) := \|A^*A - AA^*\|_2,$$

$$\mu_3(A) := (\|A\|_F^2 - \sum_{i=1}^{n} |\lambda_i|^2)^{\frac{1}{2}},$$

$$\mu_4(A) := \max_i \big| |\lambda_i| - \sigma_i \big|,$$

$$\mu_5(A) := \|H_1 - H_2\|_F \quad ,$$

$$\mu_6(A) := \min\{(\sum_{j=1}^{n} |\lambda_j - (\alpha_{p(j)} + i\beta_{q(j)})|^2)^{\frac{1}{2}},\ p,q \text{ permute } \{1,\dots,n\}\}$$

$$\mu_7(A) := \min\ \{\|U-V\|_F \,|U^*(A+A^*)\,U, V^*(A-A^*)\,V \in \mathcal{D},\ U,\ V \in \mathcal{U}\ \},$$

$$\tilde{\mu}_7(A) := \min\ \{\|U-V\|_2 \,|U^*(A+A^*)\,U, V^*(A-A^*)\,V \in \mathcal{D},\ U,\ V \in \mathcal{U}\ \},$$

$$\mu_8(A) := \min\ \{\|U-V\|_F \mid U\,A\,V^* \in \mathcal{D},\ U,\ V \in \mathcal{U}\},$$

$$\tilde{\mu}_8(A) := \min\ \{\|U-V\|_2 \mid U\,A\,V^* \in \mathcal{D},\ U,\ V \in \mathcal{U}\},$$

and for A diagonalizable

$$\mu_9(A) := \min \{\kappa_F(X)-n \mid X \in C^{n,n},\ X^{-1}A\,X = \mathrm{diag}(\lambda_i) = \Lambda\},$$

$$\tilde{\mu}_9(A) := \min \{\kappa_2(X)-1 \mid X \in C^{n,n},\ X^{-1}A\,X = \mathrm{diag}(\lambda_i) = \Lambda\}.$$

It is easy to see that each of these measures is invariant with respect to unitary transformation, and conjugate transposition, i.e.,

$$\left.\begin{aligned} \mu_i(A) &= \mu_2(U^*A\,U),\ U \\ \mu_i(A) &= \mu_i(A^*), \end{aligned}\right. \qquad i = 1,\dots,9$$

and similarly for $\tilde{\mu}_i(A)$, $i = 1,2,7,8,9$.
These measures, except for $\mu_4$, $\mu_5$ and $\mu_8$, $\tilde{\mu}_8$, are also invariant with respect to shift transformation:

$$\mu_i(A) = \mu_i(A+wI),\ w \in C\ .$$

4. <u>Comparisons between Measures of Nonnormality</u>

Comparisons between the measures $\mu_i(A)$, $\tilde{\mu}_i(A)$ are given in
<u>Theorem 2</u>. For $A \in C^{n,n}$ the following singularities hold.

(C0) $\quad \tilde{\mu}_i \leqq \mu_i \leqq \sqrt{n}\,\tilde{\mu}_i, \quad i = 1,2,7,8$ ;

(C1) $\quad \dfrac{\mu_2^2}{\sqrt{6}\,\|A\|_F} \leqq \mu_3 \leqq \left(\dfrac{n^3-n}{12}\right)^{\frac{1}{4}} \mu_2$ ;

(C2) $\quad \mu_2^2 \leqq \sqrt{2(\|A\|_F^2 + \sum_{i=1}^{n} \lambda_i{}^2)} \leqq 2\,\|A\|_F\,\mu_3$ ;

(C3) $\quad \mu_2^2 \leqq 4\,\|A\|_2\,\mu_1$ ;

(C4) $\quad \mu_2^2 \leqq 4\,\|A\|_F\,\tilde{\mu}_1 + 2\sqrt{n}\,\tilde{\mu}_1^2$ ;

(C5) $\quad \mu_1 \leqq \mu_3$ ;

(C6) $\quad \mu_4 \leqq \mu_3$ ;

(C7) $\quad \mu_3^2 \leqq 2\sqrt{n}\ \|A\|_F\ \mu_4 \leqq 2n\ \|A\|_2\ \mu_4$ ;

(C8) $\quad \|A^+\|_2^{-1}\ \mu_5 \leqq \mu_2^2 \leqq 2\ \|A\|_2\ \mu_5$ ;

(C9) $\quad \tilde{\mu}_2^2 \leqq 2\ \|A\|_2^2\ \tilde{\mu}_8$ ;

(C10) $\quad \tilde{\mu}_2^2 \leqq 8\ \|A\|_2^2\ \tilde{\mu}_7$ ;

(C11) $\quad \mu_6^2 \leqq \mu_3^2 \leqq \max\{\|A\|_F\ \mu_6,\ 2\ \|A\|_F\ \mu_6 - \mu_6^2\} \leqq 2\ \|A\|_F\ \mu_6$ ;

(C12) $\quad \tilde{\mu}_9^2\ (1+\tilde{\mu}_9)^{-1} \leqq \mu_9 \leqq \frac{1}{2}\ n\ \tilde{\mu}_9\ (1+\tilde{\mu}_9)^{-1}$,

and if all eigenvalues $\lambda_j$ of A are simple, then

$$\mu_9 \leqq \sum_{j=1}^{n} (1+(n-1)^{-1}\ \delta_j^{-2}\ \mu_3^2)^{\frac{n-1}{2}} - n\ ,$$

where $\delta_j = \min\{|\lambda_j-\lambda_i| \quad |i \neq j\}$;

(C13) $\quad \mu_2^2 \leqq 2\ \|A\|_2\ \|A\|_F \tilde{\mu}_9(2+\tilde{\mu}_9) \leqq 2\ \|A\|_F^2\ \tilde{\mu}_9(2+\tilde{\mu}_9)$,

(C14) $\quad \mu_3^2 \leqq \|A\|_F^2\ \tilde{\mu}_9(2+\tilde{\mu}_9) \leqq \|A\|_F^2\ \tilde{\mu}_9(2+\tilde{\mu}_9)$.

(C15) if $\mu_5 < \tau := \min\ \{|\sigma_i-\sigma_j|\,|\sigma_i \neq \sigma_j\}$ , then

$$\mu_8 \leqq (n+1)\ \tau^{-1}\ \mu_5;$$

(C16) if the eigenvalues $\beta_j$, $j = 1,\ldots,n$, of $(A-A^*)/(2i)$ are pairwise distinct and

$$\mu_2^2 < (6-4\sqrt{2})\ \delta\tilde{\delta},$$

where

$$\delta := \min_{\alpha_i \neq \alpha_j} |\alpha_i - \alpha_j| \ , \quad \tilde{\delta} = \min_{i \neq j} |\beta_i - \beta_j|,$$

then

$$\mu_7 \leqq \sqrt{n}(\delta\tilde{\delta} - \frac{3}{2}\mu_2^2)^{-1} \mu_2^2 \ .$$

In all these comparisons

$$\mu_i = \mu_i(A), \ i = 1,\dots,9 \text{ and } \tilde{\mu}_i = \mu_i(A), \ i = 1,2,7,8,9.$$

The comparisons in this theorem can be visualized with a directed graph, see fig. 1. There is a directed edge from node $\mu_i$ (, or $\tilde{\mu}_i$) to $\mu_j$ (, or $\tilde{\mu}_j$), $i \neq j$, iff

$$(4.1) \qquad \mu_i(A) \leqq \varphi(\mu_j(A))$$

for some monotonically continuous $\varphi$ with $\lim_{t \to 0} \varphi(t) = 0$.

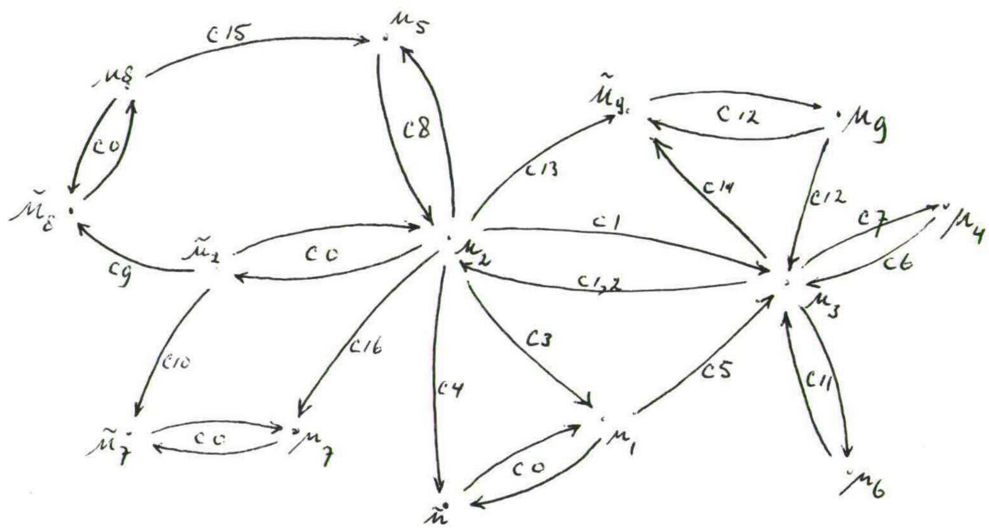


Fig. 1. The directed graph for equivalent measures.

The measures $\mu_i$, $i = 1,\dots,9$ and $\tilde{\mu}_i$, $i = 1,2,7,8,9$ are nodes in a completely connected graph, for relation 4.1 is transitive.

Proof of theorem 2. Assume $\tilde{\mu}_1 = \|A-\hat{N}\|_2$, $\mu_1 = \|A-\tilde{N}\|_F$, $\hat{N}$, $\tilde{N} \in \mathcal{N}$ .
Then

$$\tilde{\mu}_1 = \|A-\hat{N}\|_2 \leqq \|A-\tilde{N}\|_2 \leqq \|A-N\|_F = \mu_1$$

and

$$\mu_1 = \|A-\tilde{N}\|_F \leqq \|A-\hat{N}\|_F \leqq \sqrt{n}\ \|A-\hat{N}\|_2 = \sqrt{n}\ \tilde{\mu}_1.$$

The same reasoning applies to $\mu_i$, $\tilde{\mu}_i$, $i = 2,7,8$. This proofs (C0).
The first inequality in (C1) is a result of Eberlein [1], the second is a result of Henrici [2] in its original form.
The second inequality of (C2) is a consequence of

$$\sum_{i=1}^{n} |\lambda_i|^2 \leqq \|A\|_F^2 = \sum_{i=1}^{n} \sigma_i^2$$

(also known as "Schurs Lemma"), the first inequality is just a rearrangement of the inequality

$$(4.2) \qquad (\sum_{i=1}^{n} |\lambda_i|^2) \leqq \|A\|_F^4 - \frac{1}{2}\mu_2^4 ,$$

established by Kress-de Vries-Wegmann [3].
For the proof of (C3) we use that for $N \in \mathcal{N}$

$$(4.3) \qquad A^*A - AA^* = A^*(A-N) - (A-N)N^* + (A-N)^* N-A\ (A-N)^*$$

holds. Hence for any $N \in \mathcal{N}$ (, with the wellknown inequality $\|AB\|_F \leqq \|A\|_2\ \|B\|_F$,)

$$(4.4) \qquad \mu_2^2 \leqq 2(\|A\|_2 + \|N\|_2)\ \|A-N\|_F .$$

If N is such that $\mu_1 = \|A-N\|_F$ and $U^*NU = D \in \mathcal{D}$ for $U \in \mathcal{U}$ , then it is obvious from

(4.5) $\mu_1 = \|A-N\|_F = \min\{\|U^*AU-D\|_F \mid D \in \mathcal{D}, U \in \mathcal{U}\}$

that D is the diagonal part of $U^*AU$. In particular

(4.6) $\|N\|_2 = \|D\|_2 \leqq \|A\|_2$ .

As follows from the definition of the spectral norm (4.4) and (4.6) together yield (C3).
Similarly from (4.3)

(4.7) $\mu_2^2 \leqq 2(\|A\|_F + \|N\|_F)\ \|A-N\|_2$ ,

and, using in (4.7)

$$\|N\|_F \leqq \|A-N\|_F + \|A\|_F \leqq \sqrt{n}\ \|A-N\|_2 + \|A\|_F ,$$

(C4) follows.
For (C5) we consider a unitary matrix V that transforms A to upper triangular form:

(4.8) $V^*AV = \Lambda + M.$

Then

$$\|A\|_F^2 = \|\Lambda\|_F^2 + \|M\|_F^2 .$$

Hence

$$\mu_3 = \|M\|_F = \|A-V^*\Lambda V\|_F .$$

This implies $\mu_3 \geqq \mu_1$ .
(C6) is a result of Ruhe [5], while its counterpart (C7) is an easy consequence of

$$\mu_3^2 = \sum_{i=1}^{n} (\sigma_i^2 - |\lambda_i|^2) = \sum_{i=1}^{n} (\sigma_i + |\lambda_i|)(\sigma_i - |\lambda_i|)$$

$$\leq \mu_4 \sum_{i=1}^{n} (\sigma_i + |\lambda_i|) \leq \mu_4(\sqrt{n}\ (\sum_{i=1}^{n} \sigma_i^2)^{\frac{1}{2}} + \sqrt{n}\ (\sum_{i=1}^{n} |\lambda_i|^2)^{\frac{1}{2}})$$

$$\leq 2\sqrt{n}\ \|A\|_F\ \mu_4 .$$

The second inequality in (C7) follows from $\|B\|_F \leq \sqrt{n}\ \|B\|_2$ for every $B \in C^{n,n}$.

For the proof of the second inequality of (C8) we make use of the singular value decomposition

$$(4.9) \qquad A = W\ \Sigma\ V^* ,$$

where $W,\ V \in U$ and $\Sigma = \mathrm{diag}(\sigma_i)$. As

$$A = W\ \Sigma\ W^*\ W\ V^*,$$

and

$$A = W\ V^* V\ \Sigma\ V^* ,$$

we have that

$$A = H_1 U = U\ H_2 ,$$

where

$$H_1 = W\ \Sigma\ W^*,\ H_2 = V\ \Sigma\ V^* \text{ and } U = W\ V^*.$$

An easy calculation gives

$$(4.10) \qquad H_1 - H_2 = W(\Sigma Y - Y\Sigma)V^*$$

and

$$(4.11) \qquad A^*A - AA^* = W(Y\Sigma^2 - \Sigma^2 Y)V^*$$

where

$$Y = W^* V = (y_{ij}) \in \mathcal{U}.$$

Hence from (4.10)

$$(4.12) \qquad \mu_5^2 = \|H_1 - H_2\|_F^2 = \|\Sigma Y - Y\Sigma\|_F^2 = \sum_{i,j=1}^n |y_{ij}|^2 (\sigma_i - \sigma_j)^2$$

where by (4.11)

(4.13)

$$\mu_2^4 = \|A^* A - AA^*\|_F^2 = \sum_{i,j=1}^n |y_{ij}|^2 (\sigma_i^2 - \sigma_j^2)^2 = \sum_{i,j=1}^n |y_{ij}|^2 (\sigma_i - \sigma_j)^2 (\sigma_i + \sigma_j)^2 .$$

This shows

$$\mu_2^4 \leqq 4\ \sigma_1^2\ \mu_5^2 = 4\ \|A\|_2^2\ \mu_5^2 ,$$

i.e., the second inequality of (C8).

To prove the first inequality of (C8) we observe that with

$$\|A^+\|_2 = \max\{\sigma_i^{-1} \mid \sigma_i > 0,\ i = 1,\ldots,n\}$$

one has

$$|\sigma_i - \sigma_j| \leqq \|A^+\|_2\ |\sigma_i^2 - \sigma_j^2|.$$

(4.12) and (4.13) show now

$$\mu_5^2 \leqq \|A^+\|_2^2\ \mu_4^2$$

i.e., the first inequality of (C8).

For the proof of inequality (C9) we assume that $\tilde{\mu}_8 = \|U_0 - V_0\|_2$, where $U_0, V_0 \in \mathcal{U}$ and $U_0\ A\ V_0^* = D \in \mathcal{D}$ . Then

$$A^* A - AA^* = V_0^* D^* D\ V_0 - U_0^* D\ D^* U_0$$

$$= (V_0-U_0)^* D^* D\, V_0 - U_0^* D^* D(V_0-U_0).$$

Hence

$$\tilde{\mu}_2^2 \leqq 2\ \|D\|_2^2\ \|V_0-U_0\|_2 = 2\ \|A\|_2^2\ \tilde{\mu}_8 \ .$$

For the proof of (C10) we assume that $\tilde{\mu}_7 = \|U_1-V_1\|_2$ where

$$U_1^* \frac{A+A^*}{2} U_1 = M \in \mathcal{D} \quad , \quad V_1^* \frac{A-A^*}{2} V_1 = N \in \mathcal{D} \quad ,$$

with

$$U_1,\ V_1 \ \in \mathcal{U} \ .$$

So

$$A = U_1 M\, U_1^* + V_1 N\, V_1^* \ , \ A^* = U_1 M\, U_1^* - V_1 N\, V_1^* \ .$$

Let be

$$R = V_1 - U_1 \ .$$

Then

$$\frac{1}{2}\, (A^* A - AA^*) = U_1 M\, U_1 V_1^*\, N\, V_1^* - V_1 N\, V_1^* U_1 M\, U_1^* \ .$$

Hence, with $W = V_1^*\, U_1$,

$$V_1^* \cdot \frac{1}{2}(A^*A-AA^*)V_1 = WMW^*N - NWMW^*$$

$$= (W-I)MW^*N + M(W^*-I)N - NM(W^*-I) - N(W-I)MW^* .$$

Consequently,

$$\frac{1}{2}\ \tilde{\mu}_2^2 \leqq \|(W-I)MW^*N\|_2 + \|M(W^*-I)N\|_2 + \|NM(W^*-I)\|_2 + \|N(W-I)MW^*\|_2$$

$$\leqq 4 \|W-I\|_2 \|N\|_2\|M\|_2 \leqq 4 \|U_1-V_1\|_2 \|A\|_2^2 = 4 \|A\|_2^2 \tilde{\mu}_7 .$$

This proofs (C10).

Without loss of generality we may assume in the proof of the first inequality of (C11) that A is an uppertriangular matrix since both $\mu_6$ as $\mu_3$ are unitary invariants of A : $\mu_6(A) = \mu_6(U^*AU)$, $\mu_3(U^*AU) = \mu_3(A)$ for $U \in \mathcal{U}$ . So let be

$$A = \Lambda + R,$$

where

$$r_{ij} = 0, \ 1 \leqq j < i \leqq n$$

and

$$\Lambda = \mathrm{diag}(\lambda_j),$$

with

$$\lambda_j = \tau_j + i\ \nu_j, \ j = 1,\ldots,n .$$

Then

$$\left\|\frac{A+A^*}{2} - \frac{\Lambda+\Lambda^*}{2}\right\|_F^2 = \left\|\frac{1}{2}(R+R^*)\right\|_F^2 = \frac{1}{2} \|R\|_F^2 = \frac{1}{2} \mu_3^2 .$$

According to the Hoffman-Wielandt theorem [8] there exists a permutation $j \to p(j)$ of $\{1,\ldots,n\}$ such that

$$\sum_{j=1}^{n} (\tau_j-\alpha_{p(j)})^2 \leqq \left\|\frac{A+A^*}{2} - \frac{\Lambda+\Lambda^*}{2}\right\|_F^2 = \frac{1}{2} \mu_3^2 .$$

Similarly one finds for the skewhermitean part

$$\left\|\frac{A-A^*}{2} - \frac{\Lambda-\Lambda^*}{2}\right\|_F^2 = \left\|\frac{1}{2}(R-R^*)\right\|_F^2 = \frac{1}{2} \|R\|_F^2 = \frac{1}{2} \mu_3^2 .$$

For some permutation $j \to q(j)$ of $\{1,\ldots,n\}$

$$\sum_{j=1}^{n} (\nu_j - \beta_{q(j)})^2 \leqq \frac{1}{2}\mu_3^2 .$$

Consequently

$$\mu_6 \leqq \mu_3 .$$

So far as concerns the second inequality of (C11) observe

$$\mu_3^2 = \|A\|_F^2 - \sum_{j=1}^{n} |\lambda_j|^2 = \|\frac{A+A^*}{2}\|_F^2 + \|\frac{A-A^*}{2}\|^2 - \sum_{j=1}^{n} |\lambda_j|^2$$

$$= \sum_{j=1}^{n} (\alpha_j^2 + \beta_j^2 - |\lambda_j|^2).$$

Assume $\lambda_j = \tau_j + i\,\nu_j$, $j = 1,\ldots,n$. Then

$$\mu_3^2 = \sum_{j=1}^{n} (\alpha_{p(j)}^2 - \tau_j^2) + \sum_{j=1}^{n} (\beta_{q(j)}^2 - \nu_j^2),$$

where the permutations p,q realize the minimum that occur in the definition of $\mu_6$. Hence

$$\mu_3^2 = \sum_{j=1}^{n} (\alpha_{p(j)} - \tau_j)(\alpha_{p(j)} + \tau_j) + \sum_{j=1}^{n} (\beta_{q(j)} - \nu_j)(\beta_{q(j)} + \nu_j)$$

$$\leqq \mu_6 (\sum_{j=1}^{n} ((\alpha_{p(j)} + \tau_j)^2 + (\beta_{q(j)} + \nu_j)^2))^{\frac{1}{2}}$$

$$\leqq \mu_6 ((\sum_{j=1}^{n} (\alpha_j^2 + \beta_j^2))^{\frac{1}{2}} + (\sum_{j=1}^{n} (\tau_j^2 + \nu_j^2))^{\frac{1}{2}})$$

$$= \mu_6 (\|A\|_F + (\sum_{j=1}^{n} |\lambda_j|^2)^{\frac{1}{2}}) = \mu_6 (\|A\|_F + (\|A\|_F^2 - \mu_3^2)^{\frac{1}{2}}).$$

The last equation gives

$$\mu_3^2 - \|A\|_F\ \mu_6 \leqq (\|A\|_F^2 - \mu_3^2)^{\frac{1}{2}}\ \mu_6 .$$

Either

$$\mu_3^2 \leqq \|A\|_F\ \mu_6 \ ,$$

or by squaring

$$\mu_3^2 \leqq 2\ \|A\|_F\ \mu_6 - \mu_6^2 \ .$$

Hence

$$\mu_3^2 \leqq \max\ \{\|A\|_F\ \mu_6,\ 2\ \|A\|_F\ \mu_6 - \mu_6^2\} \leqq 2\ \|A\|_F\ \mu_6 \ .$$

This proofs (C11).
The results of (C12) are due to Smith [6]. There he proofs, for

$$K := \inf\{\kappa_F(X)\ |\ X^{-1}\ A\ X \in \mathcal{D}\ \ ,\ X \in C^{n,n}\}$$

and

$$k := \inf\{\kappa_2(X)\ |\ X^{-1}\ A\ X \in \mathcal{D}\ \ ,\ X \in C^{n,n}\}$$

that

$$n - 2 + k + k^{-1} \leqq K \leqq \frac{1}{2}\ n(k+k^{-1}).$$

Since

$$\mu_9 = K - n,\ \tilde{\mu}_9 = k - 1$$

this yields

$$\tilde{\mu}_9^2\ (1+\tilde{\mu}_9)^{-1} \leqq \mu_9 \leqq \frac{1}{2}\ n\ \tilde{\mu}_9(1+\tilde{\mu}_9)^{-1} \ .$$

In the same paper Smith derives, if all eigenvalues of A are simple, with

$$\delta_j = \min_{i \neq j} |\lambda_j - \lambda_i| \; .$$

that

$$K \leqq \sum_{j=1}^{n} (1+(n-1)^{-1} \delta_j^{-2} \mu_3^2)^{-1} \; ,$$

being the last inequality in (C12).

For the proof of (C13) we use the following facts:

a. If $S$ is hermitean and $X$ nonsingular, then Schurs Lemma applied to $X^{-1}S\,X$ yields

$$(4.14) \qquad \|S\|_F \leqq \|X^{-1}S\,X\|_F \; .$$

b. If $Y$ is positive definite and $G \in C^{n,n}$, then

$$(4.15) \qquad \|Y\,G\,Y^{-1} - G\|_F \leqq (\kappa_2(Y)-1)\;\|G\|_F \; .$$

This can be shown by writing the linear operator

$$L : C^{n,n} \to C^{n,n},\; L(G) := Y\,G\,Y^{-1} - G$$

in the usual vectorized form (, see Marcus-Minc, [4], p. 9): $\mathrm{vec}(L(G)) = (Y^{-1} \otimes Y - I_n \otimes I_n)\,\mathrm{vec}(G) = \tilde{L}\,\mathrm{vec}(G)$, where $\otimes$ denotes the Kronecker product. As $\tilde{L}$ is hermitean and has eigenvalues $-1 + \eta_i/\eta_j$ (, $\eta_i$ eigenvalues of $Y$) and as $\|G\|_F$ is the usual euclidean norm of the vector $\mathrm{vec}(G) \in C^{n^2}$, the result (4.15) follows.

Assume $\tilde{\mu}_9 = \kappa_2(X_1)-1$. From $A = X_1^{-1}\,\Lambda\,X_1$ we derive the following relation

$$X_1(A^*A-AA^*)X_1^{-1} = (Y\,\Lambda^*Y^{-1} - \Lambda^*)\;\Lambda - \Lambda(Y\,\Lambda^*\,Y^{-1} - \Lambda^*)$$

where

$$Y = X_1\,X_1^* > 0.$$

Using (4.14) and (4.15) yields

$$\mu_2^2 \leqq 2 \|\Lambda\|_2 \|Y\Lambda^* Y^{-1} - \Lambda^*\|_F \leqq 2 \|\Lambda\|_2 \|\Lambda^*\|_F (\kappa_2(Y) - 1)$$

$$= 2 \|\Lambda\|_2 \|\Lambda\|_F (\kappa_2^2(X_1) - 1)$$

$$= 2 \|\Lambda\|_2 \|\Lambda\|_F \tilde{\mu}_9 (2 + \tilde{\mu}_9)$$

$$\leqq 2 \|A\|_F^2 \tilde{\mu}_9 (2 + \tilde{\mu}_9).$$

For the proof of (C14) we observe that $A = X_1^{-1} \Lambda X_1$ yields

$$\|A\|_F^2 \leqq \kappa_2^2(X_1) \|\Lambda\|_F^2 .$$

Hence, with $\tilde{\mu}_9 = \kappa_2(X_1) - 1$,

$$\mu_3^2 = \|A\|_F^2 - \|\Lambda\|_F^2 \leqq (K_2^2(X_1) - 1) \|\Lambda\|_F^2 = \tilde{\mu}_9(2 + \tilde{\mu}_9) \|\Lambda\|_F^2$$

$$\leqq \|A\|_F^2 \tilde{\mu}_9(2 + \tilde{\mu}_9),$$

i.e. (C14).

In the proof of inequality (C15) we make use of two lemmata. These describe the relation between several measures of non-unitarity of matrices. For that purpose we introduce the following notations

$$\pi_1(A) := \min\{\|A-U\|_F \mid U \in \mathcal{U}\}, \quad \tilde{\mu}_1(A) := \{\|A-U\|_2 \mid U \in \mathcal{U}\},$$

$$\pi_2(A) := \min\{\|UAU^*\Delta - I\|_F \mid U \in \mathcal{U}, \Delta \in \mathcal{U} \in \mathcal{D}\}, \quad \pi_3(A) := \|A^*A - I\|_F$$

$$\pi_2(A) := \min\{\|UAU^*\Delta - I\|_2 \mid U \in \mathcal{U}, \Delta \in \mathcal{U} \cap \mathcal{D}\}, \quad \tilde{\pi}_3(A) := \|A^*A - I\|_2.$$

<u>Lemma 1</u>. $\pi_1(A) = \pi_2(A)$, $\tilde{\pi}_1(A) = \tilde{\pi}_2(A)$.

<u>Proof</u>. Let be $\pi_1(A) = \|A-V\|_F$, $V \in \mathcal{U}$. Then there exists a unitary Q and a unitary diagonal matrix $\Delta$ such that $Q V Q^* \Delta = I$. If $A = V + E$, then

$$\pi_2(A) \leqq \|Q(V+E)Q^*\Delta - I\|_F = \|QEQ^*\Delta\|_F = \pi_1(A).$$

Reservely, let be $\pi_2(A) = \|Q\ A\ Q^*\Delta - I\|_F$ with $Q \in \mathcal{U}$ and $\Delta \in \mathcal{U} \cap \mathcal{D}$. If $Q\ A\ Q^*\Delta = I + G$, then $A = Q^*(I+G)\Delta^* Q$ and $Q^*\Delta^* Q \in \mathcal{U}$. So

$$\pi_1(A) \leqq \|A - Q^*\Delta^* Q\|_F = \|Q^* G\ \Delta^* Q\|_F = \|G\|_F = \pi_2(A).$$

The same arguments yield $\tilde{\pi}_1(A) = \tilde{\pi}_2(A)$ □

<u>Lemma 2</u>. $\pi_1(A) \leqq \pi_3(A) \leqq \pi_1(A)(2+\pi_1(A))$, $\tilde{\pi}_1(A) \leqq \tilde{\pi}_3(A) \leqq \tilde{\pi}_1(A)(2+\tilde{\pi}_1(A))$.

<u>Proof</u>. Assume $A = U\ \Sigma\ V^*$, the singular value decomposition of A and let be $\Sigma = I + D$. Then $D_{ii} \geqq -1$ and

$$\|A - U\ V^*\|_F = \|U\ D\ V^*\|_F = \|D\|_F \geqq \pi_1(A).$$

Furthermore

$$\pi_3(A) = \|A^* A - I\|_F = \|V(I+D)^2 V^* - I\|_F = \|2D + D^2\|_F \geqq \|D\|_F \geqq \pi_1(A),$$

for $D_{ii} \geqq -1$. With $\pi_1(A) = \|A - V\|_F$, $V \in \mathcal{U}$ and $A = V + E$ one finds

$$\pi_3(A) = \|A^* A - I\|_F = \|(V+E)^*(V+E) - I\|_F = \|V^* E + E^* V + E^* E\|_F$$

$$\leqq 2\ \|E\|_F + \|E\|_F^2 = 2\ \pi_1(A) + \pi_1^2(A).$$

The same arguments yield $\tilde{\pi}_1(A) \leqq \tilde{\pi}_3(A) \leqq \tilde{\pi}_1(A)\ (2 + \tilde{\pi}_1(A))$ □

For the proof of (C15) let be $A = U\ \Sigma\ V^*$ a singular value decomposition of A. Then $H_1 = U\ \Sigma\ U^*$ and $H_2 = V\ \Sigma\ V^*$. Then, with $W = U^* V$ one has

$$U^*(H_1 - H_2)V = \Sigma W - W\Sigma\ .$$

Without loss of generality we may assume that equal singular values are consecutive in the diagonal of $\Sigma$. When the multiplicities of the r (, $2 \leqq r \leqq n$,) distinct singular values $\sigma_{k_1}, \ldots, \sigma_{k_r}$ are $n_1, \ldots, n_r$ resp. then we write

$$\Sigma = \begin{pmatrix} \sigma_{k_1} I_1 & 0 & & 0 \\ 0 & \sigma_{k_2} I_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \sigma_{k_r} I_r \end{pmatrix}, \quad I_j \in C^{n_j, n_j},$$

with $\sigma_{k_i} \neq \sigma_{k_j}$ for $i \neq j$. Hence $\tau = \min_{i \neq j} |\sigma_{k_i} - \sigma_{k_j}|$ . Similarly we partition W. So we obtain

$$(4.16) \qquad \mu_5^2 = \| U^*(H_1 - H_2) V \|_F^2 = \sum_{i \neq j} (\sigma_{k_i} - \sigma_{k_j})^2 \| W_{ij} \|_F^2 \geq \tau^2 \sum_{i \neq j} \| W_{ij} \|_F^2 ,$$

where $W_{ij} \in C^{n_i, n_j}$ is a block in the partitioned matrix W. (4.16) yields

$$(4.17) \qquad \sum_{i \neq j} \| W_{ij} \|_F^2 \leqq (\mu_5/\tau)^2 =: \varepsilon^2 < 1. .$$

In each diagonal block $W_{jj} \in C^{n_j, n_j}$ of W the length of each column is in the interval $[(1-\varepsilon^2)^{\frac{1}{2}}, 1]$ and the inner product of each pair of columns of each $W_{jj}$ is smaller than $\varepsilon^2$. Hence

$$(4.18) \qquad \pi_3(W_{jj}) = \| W_{jj}^* W_{jj} - I \|_F \leqq n_j \, \varepsilon, \; j = 1, \ldots, r.$$

In view of lemma 2 there exists a unitary $Q_j \in C^{n_j, n_j}$ and a unitary diagonal $\Delta_j \in C^{n_j, n_j}$ such that

$$\pi_1(W_{jj}) = \pi_2(W_{jj}) = \| Q_j^* W_{jj} Q_j \Delta_j - I \|_F \leqq \pi_3(W_{jj}) \leqq n_j \, \varepsilon, \qquad j = 1, \ldots r.$$

These matrices $Q_j$ and $\Delta_j$ are used in the formation of the block diagonal matrices

$$D_1 = \begin{pmatrix} Q_1 & 0 & \cdots & 0 \\ 0 & Q_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & Q_r \end{pmatrix}, \quad D_2 = \begin{pmatrix} Q_1\Delta_1 & 0 & \cdots & 0 \\ 0 & Q_2\Delta_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & Q_r\Delta_r \end{pmatrix}.$$

Let be

$$\tilde{U} := U\,D_1, \quad \tilde{V} := V\,D_2 .$$

Then

$$\tilde{U}^*\,A\,\tilde{V} = D_1^*\,\Sigma\,D_2 = \mathrm{diag}(\sigma_{k_j}\Delta_j) \in \mathcal{D}$$

and

$$\|\tilde{V}-\tilde{U}\|_F^2 = \|\tilde{U}^*\,\tilde{V} - I\|_F^2 = \|D_1^*\,U^*\,V\,D_2 - I\|_F^2$$

$$= \|\mathrm{diag}(Q_j^*\,W_{jj}Q_j\Delta_j - I)\|_F^2 + \sum_{i\neq j} \|W_{ij}\|^2 .$$

With (4.17) one has

$$\|\tilde{V}-\tilde{U}\|_F^2 \leqq \sum_{j=1}^{r} n_j^2\,\varepsilon^2 + \varepsilon^2 = \varepsilon^2(1+\sum_{j=1}^{r} n_j^2) \leqq \varepsilon^2(1+n^2) .$$

This means

$$\mu_8 \leqq \varepsilon\,\sqrt{1+n^2} \leqq (n+1)\,\mu_5/\tau .$$

For the proof of (C16), as can be done without loss of generality, we assume that $M := (A+A^*)/2 = \mathrm{diag}(\alpha_j)$.
When the multiplicities of the r distinct eigenvalues $\alpha_{k_1},\ldots,\alpha_{k_r}$ (, $1 < r \leqq n$,) are $n_1,\ldots,n_r$ resp. we write

$$M = \frac{A+A^*}{2} = \begin{pmatrix} \alpha_{k_1} I_1 & 0 & \cdots & 0 \\ 0 & \alpha_{k_2} I_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \alpha_{k_r} I_r \end{pmatrix} , \quad I_j \quad R^{n_j, n_j} .$$

The matrix $G = (A-A^*)/(2i)$ is partitioned in the same way and we assume that each diagonal block $G_{jj}$ has been unitarily diagonalized. Hence

$$G = \frac{A-A^*}{2} = \mathrm{diag}(\gamma_1, \ldots, \gamma_n) + \begin{pmatrix} 0 & G_{12} & \cdots & G_{1n} \\ G_{21} & 0 & \cdots & G_{2n} \\ \vdots & \vdots & & \vdots \\ G_{n1} & G_{n2} & \cdots & 0 \end{pmatrix} .$$

Then

$$(4.19) \qquad \frac{1}{4} \|A^*A - AA^*\|_F^2 = \|GM-MG\|_F^2 = \sum_{i \neq j} (\alpha_{k_i} - \alpha_{k_j})^2 \|G_{ij}\|_F^2 \geq \delta^2 \sum_{i \neq j} \|G_{ij}\|_F^2 ,$$

where $\quad \delta = \min_{i \neq j} |\alpha_{k_i} - \alpha_{k_j}| = \min_{\alpha_i \neq \alpha_j} |\alpha_i - \alpha_j| .$

Hence (4.19) implies

$$(4.20) \qquad \left( \sum_{i \neq j} \|G_{ij}\|_F^2 \right)^{\frac{1}{2}} \leqq \frac{1}{2} \mu_2^2 / \delta .$$

With the theorem of Hoffman-Wielandt [8] one derives that there exists a permutation $p$ of the eigenvalues $\beta_j$ of $G$ such that

$$\left( \sum_{j=1}^{n} (\beta_{p(j)} - \gamma_j)^2 \right)^{\frac{1}{2}} \leqq \frac{1}{2} \mu_2^2 / \delta .$$

So $\qquad |\beta_{p(j)} - \gamma_j| \leqq \frac{1}{2} \mu_2^2 / \delta$ , $j = 1, \ldots, n$, and

$$(4.21) \qquad |\gamma_i-\gamma_j| \geqq |\beta_{p(i)}-\beta_{p(j)}| - |\beta_{p(i)}-\gamma_i| - |\beta_{p(j)}-\gamma_j| \geqq \tilde{\delta} - \mu_2^2/\delta, \quad i \neq j,$$

where

$$\tilde{\delta} = \min_{i\neq j} |\beta_i-\beta_j| ,$$

the minimal distance of the distinct eigenvalues of G.
Now we investigate the distance between $e_1 = (1,0,\ldots,0)^T$ and the normalized eigenvalues of G corresponding with $\beta_{p(1)}$. For that purpose G is partitioned in the following way

$$G = \left(\begin{array}{c|c} \gamma_1 & 0^T \\ \hline 0 & \mathrm{diag}(\gamma_2,\ldots,\gamma_n) \end{array}\right) + \left(\begin{array}{c|c} 0 & g^T \\ \hline g & \tilde{G} \end{array}\right), \quad \tilde{G} \in C^{n-1,n-1} .$$

Since $\min\limits_{2\leqq i\leqq n} |\gamma_1-\gamma_i| \geqq \tilde{\delta} - \mu_2^2/\delta$ as follows from (3.21) and $\|\tilde{G}\|_F \leqq \frac{1}{2}\mu_2^2/\delta$ as follows from (4.20),

$$\min_{2\leqq i\leqq n} |\gamma_1-\gamma_i| - \|\tilde{G}\|_F \geqq \tilde{\delta} - \frac{3}{2}\mu_2^2/\delta > 0 .$$

Moreover, $\|g\|_2 \leqq \frac{1}{2}\mu_2^2/\delta$, once again as a consequence of (4.20). Applying Stewart's theorem on the sensitivity of an invariant subspace [7] we conclude that there exists a vector $p_1 \in C^{n-1}$ satisfying

$$\|p_1\|_2 \leqq 2\|g\|_2/(\tilde{\delta} - \frac{3}{2}\mu_2^2/\delta) \leqq \mu_2^2/(\delta\tilde{\delta} - \frac{3}{2}\mu_2^2)$$

such that $(1,p_1^T)^T \in C^n$ is an eigenvector of G.

The distance between $e_1$ and $v_1 := (1 + \|p_1\|^2)^{\frac{1}{2}} (1,p_1^T)^T$ equals

$$(2(1-(1+\|p_1\|_2^2)^{-\frac{1}{2}})^{\frac{1}{2}} \leqq \|p_1\|_2 \leqq \mu_2^2/(\delta\tilde{\delta} - \frac{3}{2}\mu_2^2).$$

By assumption

$$\mu_2^2/(\delta\tilde{\delta} - \frac{3}{2}\mu_2^2) < \frac{1}{2}\sqrt{2}$$

so $\|v_1-e_j\| > \|v_1-e_1\|$ for each unit vector $e_j$ with $j > 1$.
In an analogous manner one can derive the existence of eigenvalues $v_j$, $j = 2,\ldots,w$, such that $\|v_j-e_j\| < \mu_2^2/(\delta\tilde{\delta} - \frac{3}{2}\mu_2^2) < \frac{1}{2}\sqrt{2}$ .
Let be $v_1,\ldots,v_n$ the columns of $V \in U$ . Then

$$\mu_7 \leqq \|V-I\|_F \leqq \sqrt{n}\,\mu_2^2(\delta\tilde{\delta} - \frac{3}{2}\mu_2^2)^{-1}.$$

This proofs inequality (C16) and ends the proof of theorem 2. □

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